CONCERNING ZERO-DIMENSIONAL SETS IN EUCLIDEAN SPACE*

BY

R. L. WILDER

A point set M, lying in a euclidean space, E_n, of n dimensions, is said to be zero-dimensional in the Menger-Urysohn sense, if for every point P of M and every positive number ϵ there exists a separation of M into two mutually separated sets M_1 and M_2 such that M_1 contains P and the diameter of M_1 is less than ϵ.†

The present paper is intended to serve as a contribution to the study of zero-dimensional sets in E_n, particularly with reference to the relations of these sets to their complements. The property of accessibility of a point set from all sides is introduced and it is shown that in E_n $(n>1)$ all zero-dimensional sets possess this property and in E_2 are characterized by it. An example is given to show that, in the definition of accessibility used, arcs cannot be employed instead of continua. The same example shows that if M is a zero-dimensional set in E_2, then, although it is well known that M is homeomorphic with a subset of I_2, the set of all points in E_2 both of whose coördinates are irrational, there does not in general exist a one-to-one continuous transformation of E_2 into itself which carries M into a subset of I_2; i.e., M is not in general isotopic with a subset of I_2. A necessary and sufficient condition is then obtained under which M will be isotopic with a subset of I_2, and by application of this condition it is found that every punctiform F_σ in E_2 is isotopic with a subset of I_2.

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Definition. In this paper, the term ***region*** will be used to denote a simply connected, bounded domain.

It follows from a theorem of Sierpinski‡ that zero-dimensional sets are

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† For a general summary of the Menger-Urysohn dimension theory and references, see K. Menger, *Bericht über die Dimensionstheorie*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 35 (1926), pp. 113–150.

‡ W. Sierpinski, *Sur les ensembles connexes et non connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 81–95.

identical with those sets that are punctiform and homeomorphic* with linear sets. Mazurkiewicz has shown† that if a_1 and a_2 are points of a punctiform set A in E_n which is homeomorphic with a linear set, and D is a domain of E_n containing a_1, then there exists a continuum C which lies wholly in D, contains no point of A and separates a_1 from a_2. Hence, if M is a zero-dimensional set in E_2, P and Q are distinct points of M, ϵ is any positive number less than the distance from P to Q, and D is the set of all points whose distance from P is less than ϵ, there exists a set F which is a subcontinuum of $D-D\times M$ and which separates P from Q. If R is that component‡ of E_2-F determined by P, then R is a domain. Its boundary, B, is a continuum by virtue of a theorem of Brouwer;§ hence R is simply connected|| and consequently a region as defined above, its boundedness being evident. That B contains no point of M is obvious, since B is a subset of F.

If a point set M in E_2 has the property that for every point P of M and every positive number ϵ there exists a region containing P every point of which is at a distance from P less than ϵ and whose boundary contains no point of M, then it is obvious that M is zero-dimensional. We have then the following lemma:

LEMMA 1. *In order that a point set M in E_2 should be zero-dimensional it is necessary and sufficient that for every point P of M and every positive number ϵ there exist a region containing P every point of which is at a distance from P less than ϵ and whose boundary contains no point of M.*

DEFINITION. If P is a point of a point set M in E_2, then M will be said to be *locally separated* at P provided that for every positive number ϵ there exists a region containing P whose diameter is less than ϵ and whose boundary contains no point of M. If M is locally separated at all of its points, then M will be called locally separated.

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* Two sets M and N are said to be *homeomorphic* if there exists a one-to-one continuous correspondence between them. A set is called *punctiform* if it contains no continuum. (A continuum is a closed and connected point set containing more than one point.)

† S. Mazurkiewicz, *Sur un ensemble G_δ, punctiforme, qui n'est pas homéomorphe avec aucun ensemble linéaire*, Fundamenta Mathematicae, vol. 1 (1920), pp. 61–81, Theorem IV. It is clearly intended that the word "punctiforme" appear in this theorem—thus: "*A* est un ensemble *punctiforme* de $R_q \cdots$."

‡ If M is a point set and P a point of M, then that component of M determined by P is the set of all points $\{x\}$, of M, such that x and P lie in a connected subset of M.

§ Cf. L. E. J. Brouwer, *Beweis des Jordanschen Kurvensatzes*, Mathematische Annalen, vol. 69 (1910), pp. 169–175, Theorem 3.

|| Cf. R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260, Theorem 2.

DEFINITION. A sequence of regions, G, is said to *close down* on a point P if every region of G contains P and if for every positive number ϵ all but a finite number of regions of G are of diameter less than ϵ.

DEFINITION. A set of regions, G, is said to *cover* a point set M *in the Vitali sense* provided that if P is any point of M there exists an infinite sequence of regions of G closing down on P.

DEFINITION. A set of regions, G, is said to have *property* H if for every positive number ϵ there exist only a finite number of regions of G of diameter greater than ϵ.

DEFINITION. A point P of a set M in E_2 will be said to be *accessible from all sides* provided that if D is a Jordan region* whose boundary contains P, then P can be joined to any point x of D by a continuum C which lies wholly in D except for P, and such that all points of C except P and possibly x are points of $E_2 - M$. If in this definition the words "a continuum" are replaced by "an arc," the point P will be called *arcwise accessible from all sides.*

DEFINITION. If M is a set of points in E_2, then E_2 will be said to be *accessible from all sides with respect to* M if every point P of E_2 is accessible from all sides when P is added to M. An analogous definition for "arcwise accessible from all sides with respect to M" is obvious.

DEFINITION. Two sets M and N lying in spaces S and T, respectively, are said to be *isotopic* in case there exists a one-to-one continuous correspondence between S and T under which M and N correspond to one another. Obviously S and T can be the same space.

DEFINITION. If P is a point of a point set M in E_2, then M will be said to be *simply locally separated* at P provided that for every positive number ϵ there exists a region containing P whose diameter is less than ϵ and whose boundary is a simple closed curve which contains no point of M. If M is simply locally separated at all of its points, then M will itself be called simply locally separated.

LEMMA 2. *If M is a zero-dimensional set, and P is a point not belonging to M, then $M+P$ is zero-dimensional.*

This lemma is an immediate consequence of a result due to Urysohn,† to the effect that the set of points at which a set T is of dimension $\geqq 0$ is dense in itself.

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* A Jordan region is a bounded domain complementary to a simple closed curve.

† P. Urysohn, *Sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), pp. 30–137, and vol. 8 (1926), pp. 225–359. See vol. 8, pp. 272–273.

THEOREM 1. *If M is a zero-dimensional set in E_2 there exists a sequence of regions G whose boundaries contain no points of M, which cover M in the Vitali sense, and such that G has property* H.*

Consider first a bounded zero-dimensional set M. Let M' denote the set composed of M together with all its limit points. By Lemmas 1 and 2, if P is any point of M', there exists a sequence of regions, $G(P)$, closing down on P whose boundaries contain no point of M. Let G' be the collection of all regions belonging to sequences of the type $G(P)$. By the Borel Theorem there exists a finite set, G_1, of regions of G' covering M'. From G' omit all regions of diameter >1 and call the resulting set of regions G_1'. As G_1' covers M', there exists a finite subset, G_2, of G_1', which covers M'. From G_1' omit all regions of diameter $>\frac{1}{2}$ and call the resulting set of regions G_2'. In general, if G_n' consists of the set of all regions of G' of diameter $\leqq 1/n$, there exists a finite subset, G_{n+1}, of G_n', which covers M'.

Let $G=\sum_1^\infty G_n$. Then if ϵ is any positive number, only a finite number of regions of G are of diameter $>\epsilon$, and furthermore, since for every point P of M there exists a region of G_n $(n=1, 2, 3, \cdots)$ covering P, G covers M in the Vitali sense.

Since every unbounded set in E_2 is the sum of a denumerable collection of bounded sets, the set M, if unbounded, is the sum of a sequence of bounded sets $M_1, M_2, M_3, \cdots$. The set G' can be selected as before; in general G_n' can consist of all regions of G' of diameter $\leqq 1/n$ and G_{n+1} can be a finite set of regions of G_n' covering the set $\sum_1^n M_i$ together with its limit points.

THEOREM 1a. *If M is a simply locally separated set in E_2, there exists a set of Jordan regions G whose boundaries contain no point of M, which cover M in the Vitali sense, and such that G has property* H.

THEOREM 2. *If M is a zero-dimensional set in E_2, then E_2 is accessible from all sides with respect to M.*

Let D be a Jordan region and P a point on the boundary, B, of D. Let Q be any point of D. There exists a simple closed curve J which contains P and Q and which lies, except for P, wholly in D. The curve J is the sum of two arcs, t_1 and t_2, which have in common only the points P and Q.

Let z be a point of R, the region bounded by J, and let t be an arc whose end points are P and Q, which contains z, and which lies, except for P and Q, wholly in R. Let M_1 be the set of points common to M and $J-(P+Q)$.

* Note added in proof-reading: Since I have recently shown (in my paper *Concerning the Phragmen-Brouwer Theorem*, presented to the American Mathematical Society, December 27, 1928) that the theorem of Brouwer referred to above extends to n dimensions $(n>2)$, it is clear that Lemma 1 and consequently Theorem 1 are true in E_n, for $n>2$.

As M is zero-dimensional, there exists, by Theorem 1, a set of regions G which covers M in the Vitali sense and has property H, and such that the boundaries of regions of G contain no points of M.

If x is any point of M_1, there exists a region of G, $g(x)$, which contains x and such that $g(x)$ together with its boundary lies wholly in D and contains no point of t. The set of all points contained in regions of the type $g(x)$, together with their boundaries, denote by T. That the set of points $T+J$ is a continuum C is easily shown from the properties of G.

Let that connected domain complementary to C, which contains z, be denoted by $D(z)$. The boundary, F, of $D(z)$ is a subset of C and that it contains P and Q is easily seen from the fact that all of t, except P and Q, lies in $D(z)$. That F is a continuum follows from the Brouwer theorem referred to above, and that it contains no point of M except possibly P and Q is obvious. Hence F is a continuum which contains P and Q, contains no point of M except possibly P and Q, and lies, except for P, wholly in D.

THEOREM 3. *If M is a zero-dimensional set in E_2 and P and Q are any two points of E_2, then P and Q can be joined by a continuum K every point of which, except possibly P and Q, is in E_2-M; and indeed, if J is any simple closed curve enclosing both P and Q, K may be selected so as to lie entirely within J.*

From the above it is evident that whereas, by a theorem of Sierpinski,* the complement of a punctiform set in E_2 is connected im kleinen,† the complement of a set having the stronger property of zero-dimensionality is strongly connected im kleinen.

If, when considering space $E_n (n>2)$, we define accessibility from all sides as above, except that Jordan regions are replaced by bounded domains complementary to n-dimensional spheres, the above results are easily extended to higher spaces. Thus, we have the following theorem:

THEOREM 4. *If M is a zero-dimensional set in $E_n (n>1)$ then all points of E_n are accessible from all sides with respect to M, and the complement, E_n-M, is strongly connected im kleinen.*

I shall merely indicate how the proof is given for E_3. Let S be a sphere, P

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* W. Sierpinski, *Sur un ensemble punctiforme connexe*, Fundamenta Mathematicae, vol. 1 (1920), pp. 7-10.

† A set M is called connected im kleinen provided that if P is any point of M and ϵ is any positive number, there exists a positive number ρ such that if Q is a point at a distance from P less than ρ, there exists a connected subset N of M containing both P and Q every point of which is at a distance from P less than ϵ. If N can always be taken to be a continuum, then M is called strongly connected im kleinen.

a point on S and Q a point within S. Let T be any plane passing through P and Q. The intersection of S with T is a circle C, and with M is a point set m. The set m is zero-dimensional, and by Theorem 3 there exists a continuum K lying entirely within C, on T, and containing P and Q but no points of m except possibly P and Q. The rest of the proof should be obvious.

THEOREM 4a. *If, in $E_n(n>1)$, D is a connected domain and M is a zero-dimensional set and P and Q are distinct points of D, then there exists, in D, a continuum C which contains P and Q but which contains no point of M, except possibly P and Q.*

I shall indicate the proof for E_3. (For E_2 use Theorem 3 and the notion of simple chain indicated below.) Every point x of D is the center of a sphere S_x which lies wholly in D. Let G denote the collection of all such spheres. Then there exists, from P to Q, a simple chain, $S_1, S_2, \cdots, S_k$, of spheres* of the collection G. For each $i(i=1, 2, \cdots, k-1)$ let P_i denote a point of E_3-M common to S_i and S_{i+1}. By passing a plane through P_i and P_{i+1} and proceeding as in the proof of Theorem 4, it can be shown that there exists a continuum C_{i+1} which lies wholly in S_{i+1}, contains P_i and P_{i+1}, but no point of M. Similar continua C_1 and C_k can be obtained, where C_1 joins P and P_1 in S_1, and C_k joins P_{k-1} and Q in S_k. The continuum $C=\sum_1^k C_i$ fulfills the condition stated in the theorem.†

COROLLARY. *In $E_n(n>1)$ the complement of a zero-dimensional set is strongly connected.*

THEOREM 5. *In order that a set in E_2 should be zero-dimensional it is necessary and sufficient that it should be accessible from all sides.*

That the condition is necessary follows from Theorem 2.

The condition is also sufficient. Let P be any point of M. I shall show that M is locally separated at P. If ϵ is any positive number, let C_1, C_2, C_3, and C_4 be circles with centers at P with radii $\epsilon/4$, $\epsilon/2$, $3\epsilon/4$, and ϵ, respectively. On a radius of C_4, let the intersections with the circles $C_i(i=1, 2, 3, 4)$ occur in the order $Plkji$, and on the radius diametrically opposite let a and b be

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* If "region" be replaced by "sphere," the definition of simple chain and Theorem 10 as given on pp. 134–135 of R. L. Moore's *Foundations of plane analysis situs* (these Transactions, vol. 17 (1916), pp. 131–164) will suffice for reference here.

† It was shown by Urysohn (loc. cit., vol. 8, p. 355) that if M is an F_σ of dimension $<n-1$ in $E_n(n>1)$ and D is a connected domain, then $D-D\times M$ is strongly connected. (As a matter of fact, as I have pointed out in my paper *Concerning a theorem of J. R. Kline*, not yet published, $D-D\times M$ is *arcwise* connected.) Theorem 4a shows that for the case $n=2$ the restriction that M be an F_σ is unnecessary.

the intersections with C_1 and C_2, respectively. On C_1 let m, n, o be points in the order (counter-clockwise) $lmnoa$. Extend the radius Pn of C_1 to meet C_2 at c; and the radius Pm to meet C_2 and C_3 at d and q, respectively. On C_2 let f and e be selected so that the order (counter-clockwise) $kfedcb$ is obtained. Let a radius of C_4 through f cut C_3 and C_4 in g and h, respectively, and a radius through e meet C_3 in s. Denoting straight line intervals by brackets, and arcs of circles by parentheses, define simple closed curves J_1 and J_2 as follows:

$$J_1 = [Pab] + (bcdef) + [fgh] + (ih) + [ijkl] + (lmno) + [oP],$$

(ih) being so chosen that J_1 encloses points of $[nc]$;

$$J_2 = [Pmdq] + (qjgs) + [se] + (efkbc) + [cnP].$$

That arc jg on C_3 which does not contain s forms, with the portion $[ji]+(ih)+[hg]$ of J_1, a simple closed curve J_3. Let Q_1 be a point interior to J_3, not belonging to M. As Q_1 is interior to J_1, there exists a continuum K_1 containing Q_1 and P, and lying, except for P, wholly interior to J_1, and containing, except for P, only points of E_2-M. Let

$$J_4 = [fg] + (gs) + [se] + (ef),$$

where (gs) does not contain j, and (ef) does not contain k, and let Q_2 be a point of E_2-M interior to J_4. As Q_2 is also interior to J_2, there exists a continuum K_2 containing Q_2 and P, lying, except for P, wholly interior to J_2, and containing, except for P, only points of E_2-M.

The continuum K_1 contains a continuum T_1 which lies wholly within or on the boundary of the annular domain bounded by C_1 and C_3, and contains points on both C_1 and C_3.* The points of T_1 on C_3 lie on the arc jg of J_3, and the points of T_1 on C_1 are on that arc ao of C_1 which does not contain n. The continuum K_2 contains a continuum T_2 which lies wholly within or on the simple closed curve J_5 defined as follows:

$$J_5 = [fg] + (gjq) + [qdm] + (mn) + [nc] + (cbkf),$$

where (mn) does not contain a, and such that T_2 has points on both $[fg]$ and (mn).

The continuum $T=T_1+T_2$ does not contain P and hence is a subset of E_2-M. If D is that complementary domain of T determined by P, then D is a region whose boundary is a subset of T, and such that D contains no point on or exterior to C_4. The diameter of D is less than ϵ, and hence M is locally separated at P.

* Cf. Anna M. Mullikin, *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162, Theorem 1.

If a set M in E_2 is accessible from all sides, then by Theorem 5 it is zero-dimensional; and hence, by Theorem 2, E_2 is accessible from all sides with respect to M and, by Theorem 3, $E_2 - M$ is strongly connected im kleinen. Hence the following corollary:

COROLLARY. *If a set M in E_2 is accessible from all sides, then E_2 is accessible from all sides with respect to M, and $E_2 - M$ is strongly connected im kleinen.*

Thus in E_2 accessibility from all sides of a set, and accessibility of E_2 from all sides with respect to that set, are equivalent properties.

That ordinary accessibility of points of M, in the sense that if P is a point of M and Q a point of $E_2 - M$ there exists a continuum K containing P and Q and such that $K \times M = P$, is not sufficient, even where M is punctiform and totally disconnected, to insure the zero-dimensionality of M is shown by the example of a quasi-connected point set in my paper *A set which has no true quasi-components and which becomes connected upon the addition of a single point.** The set described in this paper is accessible by arcs.

That in the definition of accessibility from all sides used above, arcs cannot be employed instead of unrestricted continua will be shown by the example given in §2 of a zero-dimensional set which is not simply locally separated.

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It is known† that the set, I_2, of all points in E_2 both of whose coördinates are irrational is homeomorphic with the set, I_1, of points in E_1 whose abscissas are irrational. Sierpinski has shown that zero-dimensional sets are homeomorphic with linear sets and hence with subsets of I_1.‡ Hence every zero-dimensional set is homeomorphic with a subset of I_2. This suggests the question: Is a zero-dimensional set in E_2 isotopic with a subset of I_2? It is to be noticed, first, that any subset of I_2 is simply locally separated as defined above, and that any set isotopic with a subset of I_2 must accordingly be simply locally separated. Hence I shall first show that the above question cannot be answered affirmatively by giving an example of a zero-dimensional set in E_2 which is not simply locally separated at any point.

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* Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 423–427.

† Cf. M. Fréchet, *Les dimensions d'un ensemble abstrait*, Mathematische Annalen, vol. 68 (1910), pp. 145–168. See especially p. 154.

‡ Cf. Menger, *Bericht über die Dimensionstheorie*, loc. cit., pp. 125–126, and references given therein.

Consider, first, a set M constructed as follows: Let T be a unit square, and for each positive integer n divide T into n^2 equal squares, letting T_n denote the corresponding set of n^2 squares. For each square, t, of T_n, construct a continuum $K(t)$ of which every subcontinuum is indecomposable* and which lies wholly within t except that it contains four points on t, these being the mid-points of the four sides of t. For each n let M_n denote the set of points obtained by adding together the point sets $K(t)$ for all squares t of T_n. Let $M=\sum_1^\infty M_i$. This set, M, was first constructed by R. L. Moore† as an example of a connected and connected im kleinen point set which contains no arc. The point set which I wish now to consider is the complement of M within and on T. Denote this set by N.

The point set N is a zero-dimensional G_δ‡ and is everywhere dense in T and its interior. For if P is a point of N within T and C is a circle with center at P, there exists a positive integer n such that not only does a square t of T_n, within or on which P lies, lie wholly within C, but also all those squares of T_n adjacent to t. The eight continua K_t constructed relative to T_n in these adjacent squares form a continuum F. That complementary domain D of F which contains P is a region which lies wholly interior to C and whose boundary contains no point of N, being a subset of F and hence of M. Then N is locally separated at P. In case P is on T it can be shown in a similar way that N is locally separated at P. That N is everywhere dense in T and its interior is evident since every arc interior to T contains points of N. That N is a G_δ is evident since M is an F_σ. But it is obvious that N is not simply locally separated at any point and consequently N is not isotopic with I_2.

By an extension of this example there can be obtained an example of a zero-dimensional set which is dense everywhere in E_2, but which is not isotopic with I_2. This comment is of interest, perhaps, in that it relates to the analogue, for punctiform uncountable sets, of a theorem first given by Fréchet§ and later by Urysohn‖ to the effect that all denumerable sets dense in E_n are isotopic. If there exists any analogue of this theorem for uncountable sets, the example just indicated shows that zero-dimensionality is not a sufficient condition for isotopism of such sets.

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* Cf. B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fundamenta Mathematicae, vol. 3 (1922), pp. 247–286.

† R. L. Moore, *A connected and regular point set which contains no arc*, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 331–332.

‡ A G_δ is a set of points common to a denumerable infinity of open sets.

§ M. Fréchet, loc. cit., p. 159.

‖ P. Urysohn, *Sur les multiplicités Cantoriennes*, loc. cit., vol. 7, pp. 83 ff. Urysohn states that Fréchet's proof seems to him insufficient.

I shall now proceed to establish a condition which characterizes those sets which are isotopic with subsets of I_2.

THEOREM 6. *Let M be a point set in E_2. Then in order that M should be simply locally separated, it is necessary and sufficient that E_2 be arcwise accessible from all sides with respect to M.*

The condition is necessary. Let M be a simple locally separated set in E_2, and let P be any point of E_2. Let J be any simple closed curve containing P. Let D be the bounded domain complementary to J, and let Q be any point of D.

There exists a simple closed curve K which contains P and Q, and which lies, except for P, wholly in D. Denote the interior of K by R. There exists an arc t which has P and Q as end points, and which lies, except for these two points, entirely in R.

Denote the point set $M\times[K-(P+Q)]$ by M_1.

Since M is simply locally separated, there exists, by Theorem 1a, a sequence, G, of Jordan regions, which covers M in the Vitali sense and has property H, and such that the boundaries of the regions of G contain no points of M.

If x is a point of M_1, there exists a region g_x of the collection G which contains x and such that g_x' lies wholly in D and contains no point of t. The set of all points which lie in regions of the type g_x together with their boundaries denote by T. The set of points $T+K$ is a continuum C. Furthermore, C is a continuous curve. To show this, I shall employ Sierpinski's characterization* of a continuous curve; i.e., a bounded continuum N is a continuous curve provided that for every positive number e, N is the sum of a finite collection of continua each of which is of diameter less than e.

Let e be any positive number. Then K, being a continuous curve, is the sum of a finite collection of continua, $C_1, C_2, \cdots, C_n$, each of which is of diameter less than $e/4$. Since G covers M in the Vitali sense, there is only a finite number of regions of G of diameter $>e/4$. Hence those regions of G which constitute part of T and are of diameter $>e/4$ are finite in number, and as a Jordan region together with its boundary forms a continuous curve, each of them together with its boundary is the sum of a finite number of continua of diameters less than e; denote the set of all such continua by G_1. If g_x is a region which constitutes part of T and is of diameter less than $e/4$, and if C_i $(1\leqq i\leqq n)$ is a part of K which contains a point of g_x', then C_i,

* W. Sierpinski, *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fundamenta Mathematicae, vol. 1 (1920), pp. 44–60.

together with all such regions g_z and their boundaries forms a continuum C_i'. That C_i' is of diameter less than e is obvious. Therefore C is the sum of a finite collection of continua, viz., C_1', C_2', $\cdots$, C_n', and the continua of the collection G_1, all of which are of diameter less than e, and hence is a continuous curve.

If z is an interior point of t, denote by D_z that complementary domain of C determined by z. Denote the outer boundary of D_z by B. By a theorem due to R. L. Moore,* B is a simple closed curve. That B contains P and Q is easily seen, and clearly B lies wholly in D, except for P, and contains no point of M except possibly P and Q. As B is the sum of two arcs whose end points are P and Q, the condition stated in the theorem is proved necessary.

To show that the condition stated in the theorem is sufficient a modification of the proof of Theorem 5 may be employed.

THEOREM 6a. *In order that a point set in E_2 should be simply locally separated it is necessary and sufficient that it be arcwise accessible from all sides.*

COROLLARY. *If a set M in E_2 is arcwise accessible from all sides, then E_2 is arcwise accessible from all sides with respect to M, and E_2-M is arcwise connected im kleinen; and if P and Q are points of a connected domain D in E_2 there is an arc from P to Q which lies wholly in D and contains no point of M except possibly P and Q.*

The proof of the latter part of this corollary can be obtained by use of the simple chain idea (see proof of Theorem 4a), the arcwise accessibility of E_2 with respect to M, and of the fact that if C is a circle enclosing two points A and B there exists a simple closed curve within C which contains A and encloses B.

THEOREM 7. *In E_2, let M be a point set and let I_2 denote the set of all points both of whose coördinates are irrational. Then in order that M should be isotopic with a subset of I_2, it is necessary and sufficient that it be simply locally separated.*

That the condition stated in the theorem is necessary is obvious, since I_2 is itself simply locally separated.

To prove the condition sufficient, I shall proceed by methods based on R. L. Moore's *Concerning a set of postulates for plane analysis situs.*† I shall therefore first state a series of lemmas which are adaptations, as signified

* *Concerning continuous curves in the plane*, loc. cit., Theorem 4. That the boundary of D_z is itself a continuous curve is demonstrated by Moore in the proof of Theorem 4.

† These Transactions, vol. 20 (1919), pp. 169–178.

in each case, of Moore's theorems B-G. It will be understood that when a theorem is designated by letter it is one of Moore's theorems, and that when a theorem is designated by number it refers to a theorem in the present paper. Wherever a proof is not indicated it will be understood that the proof is only a slight modification of Moore's proof with the use of Theorem 6.

LEMMA 3 (Adaptation of Theorem B). *If J and L are two simple closed curves and A and B are two distinct points of $J\times(E_2-M)$ each of which is either not on L at all or on some segment that is common to J and L, then there exists an arc from A to B which lies entirely in E_2-M and, except for its end points, within J, and has not more than a finite number of points in common with L.*

The proof of Lemma 3 is similar to that of Theorem B, except that use is made of Theorem 6 (a) in establishing the existence of the arc AB in E_2-M, in the first sentence of the proof as given by Moore, and (b) in establishing the existence of the arcs $A_nZ_nB_n$ of lines 10–12, page 172, so that these have, in addition to the properties outlined there, the further property that they lie in E_2-M.

LEMMA 4 (Adaptation of Theorem C). *If the closed curve g lies in E_2-M and has only a finite number of points in common with the closed curve $ABCDA$ and does not contain A, B, C, or D, then the interior of $ABCDA$ can be divided by double ruling* such that* (1) *the arcs of this ruling lie in E_2-M,* (2) *the arcs of one of its single rulings are parallel to AB and CD and those of the other are parallel to AD and BC, and* (3) *the subdivisions of $ABCDA$ made by this ruling are such that the interior of each one of them is either wholly within or wholly without g.*

LEMMA 5 (Adaptation of Theorem D). *If $ABCDA$ is a simple closed curve and G is a set of simple closed curves which lie wholly in E_2-M and each point on or within $ABCDA$ is within some curve of the set G, then the interior of $ABCDA$ can be divided by a double ruling such that* (1) *the arcs of this ruling lie wholly in E_2-M,* (2) *the arcs of one of its single rulings are parallel to AB and CD and those of the other are parallel to AD and BC, and* (3) *the subdivisions of the interior of $ABCDA$ formed by these rulings are such that each lies within some curve of the set G.*

LEMMA 6 (Adaptation of Theorem E). *If $ABCDA$ is a simple closed curve there exist two sets of arcs, α_1 and α_2, such that* (1) *each arc of α_1 lies wholly within $ABCDA$ except that its end points are on AB and CD,* (2) *each arc of*

* For definitions see Moore's paper, loc. cit.

α_2 lies wholly within ABCDA except that its end points are on BC and DA, (3) *$\alpha_i (i=1, 2)$ is the sum of two collections of arcs, a_i and b_i, such that* (i) *if d is an arc of b_i, then d is the sequential limiting set of a sequence of arcs of the set a_i and a_i is a denumerable set,* (ii) *the arcs of a_i contain no points of M,* (4) *each point on ABCDA, with the exception of A, B, C, and D, is an end point of either just one arc of α_1 or just one arc of α_2,* (5) *through each point within ABCDA there is just one arc of α_1 and just one arc of α_2,* (6) *each arc of α_1 has just one point in common with each arc of α_2.*

The proof of Lemma 6 is the same as that of Theorem E, the curves in each set β_n being selected in $E_2 - M$, however.

LEMMA 7 (Adaptation of Theorem F). *There exists a denumerably infinite sequence of simple closed curves $J_1, J_2, J_3, \cdots$ such that every point of E_2 lies within at least one of them and such that for every n, J_{n+1} encloses J_n, and J_n lies wholly in $E_2 - M$.*

The proof of Lemma 7 is either a modification of the proof of Theorem F or an application of Theorem 6 to a series of annular domains.

LEMMA 8 (Adaptation of Theorem G). *There exist in E_2 two sets, G_1 and G_2, of open curves such that* (1) *through each point there is just one curve of G_1 and just one curve of G_2,* (2) *each curve of G_1 has just one point in common with each curve of G_2,* (3) *$G_i (i=1, 2)$ consists of two sets of open curves G_{i1} and G_{i2} such that* (a) *no curve of the set G_{i1} contains a point of M,* (b) *the set G_{i1} is denumerable,* (c) *every curve of the set G_{i2} is the sequential limiting set of a sequence of curves of the set G_{i1}.*

To establish the sufficiency of the condition stated in Theorem 7, proceed as follows on the basis of Lemma 8: Let g_1 and g_2 be particular open curves of the collection G_{11} and G_{21}, respectively. Between the points of g_1 and the points of the x-axis there is a one-to-one continuous correspondence in which the intersections of g_1 with curves of the set G_{21} correspond to points on the x-axis whose abscissas are rational and the intersections of g_1 with curves of the set G_{22} correspond to points on the x-axis whose abscissas are irrational, and in which the intersection of g_1 with g_2 corresponds to the point of the x-axis whose abscissa is zero. Similarly, between the points of g_2 and the points of the y-axis there exists a one-to-one continuous correspondence in which the intersections of g_2 with the curves of the collection G_{11} correspond to the points of the y-axis whose ordinates are rational and the intersections with the curves of the collection G_{12} correspond to the points whose ordinates are irrational and in which the intersection of g_1 and g_2 corresponds to the point whose ordinate is zero.

Let Σ_1 denote the ordinary cartesian system of coördinates, and let Σ_2 denote a system of coördinates defined as follows: If P is any point of E_2, the intersection of that curve of the collection G_2 which contains P with g_1 corresponds, in the correspondence outlined above, to a point of the x-axis whose abscissa is, say, x_2; and the intersection of that curve of G_1 which contains P with g_2 corresponds to a point of the y-axis whose ordinate is y_2; then the coördinates of P in the system Σ_2 will be (x_2, y_2). Then there exists a one-to-one continuous transformation of E_2 into itself in which two points correspond if and only if their respective coördinates in the two systems Σ_1 and Σ_2 are identical, each to each. Since points of M lie only on curves of the sets G_{21} and G_{22}, it is evident that these points correspond to points both of whose coördinates are irrational in the system Σ_1; i.e., the points of M correspond to a subset of I_2.

As a consequence of Theorems 6a and 7 we have

THEOREM 7a. *In order that a set of points in E_2 should be isotopic with a subset of I_2, it is necessary and sufficient that it be arcwise accessible from all sides.*

THEOREM 8. *In E_2, let M be a punctiform F_σ;* then E_2 is arcwise accessible from all sides with respect to M.*

In my paper *Concerning a theorem of J. R. Kline*, I have shown that if, in E_2, D is any connected domain and N is a punctiform F_σ, then the set $D-D\times N$ is arcwise connected. Accordingly, if J is any simple closed curve and P is a point of J, and Q is a point of the region, R, bounded by J, it is easy to prove that there exists an arc from P to Q which lies, except for P, and possibly Q, wholly in the set $D\times(E_2-M)$.

As a consequence of Theorems 6 and 8 we have

THEOREM 9. *In E_2 every punctiform F_σ is simply locally separated.*

As a consequence of Theorems 7 and 9 we have

THEOREM 10. *In E_2 every punctiform F_σ is isotopic with some subset of I_2, the set of all points both of whose coördinates are irrational.*

3. Some problems

1. Under what conditions is a zero-dimensional set in E_n $(n>2)$ isotopic with a subset of I_n, the set of points all of whose coördinates are irrational?

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* A set is an F_σ if it is the sum of a denumerable collection of closed sets. Cf. F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, pp. 304 ff.

2. If M is a punctiform F_σ in E_n $(n>2)$, is M isotopic with some subset of I_n?

3. When is a zero-dimensional set M in E_n $(n>2)$ arcwise accessible from all sides, and when is E_n arcwise accessible from all sides with respect to M? Are the two properties equivalent?

4. If M is a zero-dimensional set in E_n $(n>2)$ under what conditions is the complement of M arcwise connected? More generally, if D is a connected domain, when is $D-D\times M$ arcwise connected; when are any two points of D, say P and Q, the end points of an arc of D which contains no point of M, except possibly P and Q?

Problems 3 and 4 are obviously closely related, for it is easy to show that if any set M (zero-dimensional or not) is such that E_n is arcwise accessible from all sides with respect to M, and D is a connected domain and P and Q are points of D, then D contains an arc from P to Q which contains no point of M except possibly P and Q.

5. What is the maximum dimension of a point set which is accessible from all sides in E_n?

The solution of Problem 5 for $n=2$ is obviously given by Theorem 5.

6. Let M be an arbitrary point set and D a connected domain in E_n. Is there a more general condition than that given by Urysohn (see footnote accompanying Theorem 4a) sufficient to ensure the *strong* connectivity of $D-D\times M$?

University of Michigan,
Ann Arbor, Mich.